Section 12.7

Polar, Cylindrical, and Spherical Coordinates

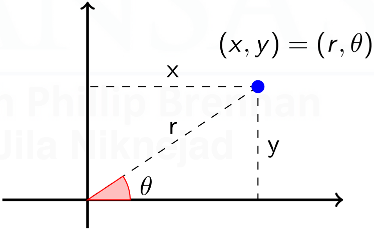
Prelecture Review Video

1 Polar Coordinates Review

Polar and Cartesian Coordinates

We usually use **Cartesian coordinates** (x, y) to represent a point in a plane. However, **polar coordinates** (r, θ) are more convenient for dealing with circles, arcs, and spirals.

- $\boldsymbol{r}$ represents the distance of a point from the origin.
- $\boldsymbol{\theta}$ is the angle in standard position (measured counterclockwise from the positive x-axis).
- It is possible that r is negative. In this case, $(-r, \theta) = (r, \theta + \pi)$.



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Polar and Cartesian Coordinates

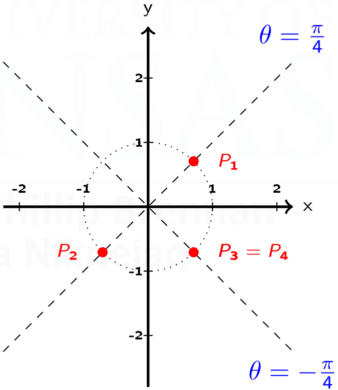
Example 1: Plot the points given in polar coordinates:

$$P_1 = \left(1, \frac{\pi}{4}\right)$$

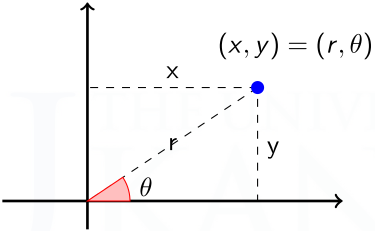
$$P_2 = \left(-1, \frac{\pi}{4}\right)$$

$$P_3 = \left(1, \frac{7\pi}{4}\right)$$

$$P_4 = \left(1, -\frac{\pi}{4}\right)$$



Polar and Cartesian Coordinates: Conversion



Conversion from polar to Cartesian coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Conversion from Cartesian to polar coordinates:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

2 Cylindrical Coordinates

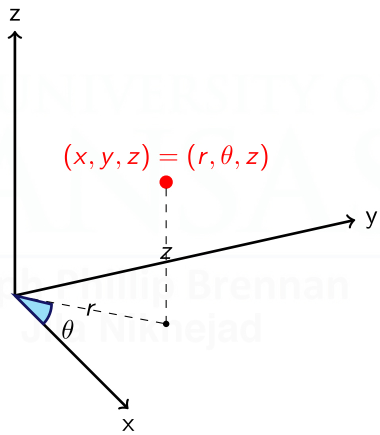
Cylindrical Coordinates in 3-Space

The **cylindrical coordinates** of a point P in three-space are

$$(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$$

where:

- r and θ are the polar coordinates of the projection of P onto the xy-plane;
- z is the same as in Cartesian coordinates.
- In cylindrical coordinates, we usually assume $r \geq 0$.



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Converting Between Cylindrical and Cartesian

Converting between cylindrical and Cartesian coordinates in 3-space is essentially the same as converting between polar and spherical coordinates in 2-space.

- Conversion from cylindrical coordinates to Cartesian coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

- Conversion from Cartesian coordinates to cylindrical coordinates:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x} \qquad z = z$$

Converting Between Cylindrical and Cartesian

Example 2: Convert the cylindrical point $(r, \theta, z) = (2, -\pi/4, 1)$ to Cartesian coordinates:

Solution:
$x = r\cos(\theta) = 2\cos\left(-\frac{\pi}{4}\right) = \sqrt{2}$ $\qquad$ $y = r\sin(\theta) = 2\sin\left(-\frac{\pi}{4}\right) = -\sqrt{2}$

Cartesian coordinates: $(\boldsymbol{\sqrt{2}}, \boldsymbol{-\sqrt{2}}, 1)$.

Example 3: Convert the Cartesian point $(x, y, z) = \left(-2,\ 2\sqrt{3},\ 1\right)$ to cylindrical coordinates:

Solution: $\qquad r = \sqrt{x^2 + y^2} = 4 \qquad\qquad \tan(\theta) = \dfrac{y}{x} = -\sqrt{3}$

Here (x, y) is in quadrant II, so $\theta = \frac{2\pi}{3}$.

Cylindrical coordinates: $(4, \boldsymbol{\frac{2\pi}{3}}, 1)$.

Converting Between Cylindrical and Cartesian

Example 4: Describe the surface given by the equation in cylindrical coordinates $z = \sqrt{r^2 + c^2}$.

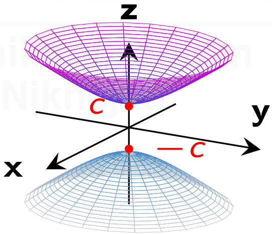
Solution: In Cartesian coordinates, the equation becomes

$$z = \sqrt{x^2 + y^2 + c^2} \qquad \text{or} \qquad x^2 + y^2 = z^2 - c^2.$$

This is the upper piece of the hyperboloid of two sheets.

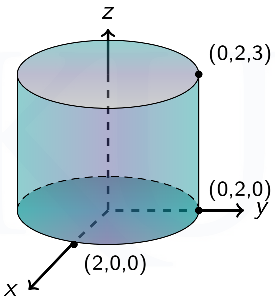
The original equation only describes the top sheet.

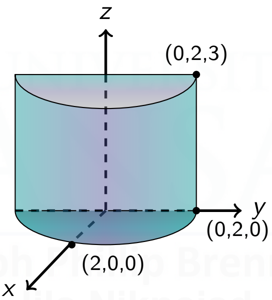
The bottom sheet is $z = -\sqrt{r^2 + c^2}$.



Converting Between Cylindrical and Cartesian

Example 5: Describe the solid regions shown below, using inequalities in cylindrical coordinates.





Link

Solution:

(a) $0 \leq r \leq 2 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq z \leq 3$

(b) $0 \leq r \leq 2 \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \qquad 0 \leq z \leq 3$

3 Spherical Coordinates

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Spherical Coordinates in 3-Space

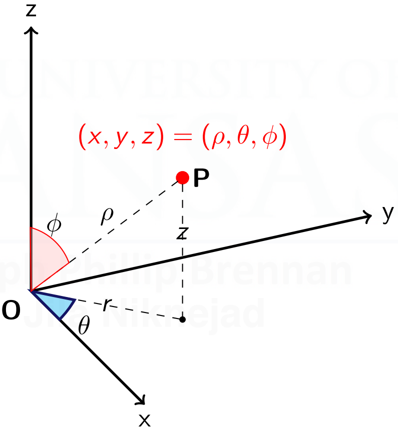
The **spherical coordinates** of a point P in three-space are

$$(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi})$$

where:

- ρ is the distance from P to the origin O
- θ is the same as in cylindrical coordinates
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$

  (so $0 \leq \phi \leq \pi$)



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Spherical Coordinates in 3-Space

- The equation $\rho = R$ defines a sphere of radius R.
- The ball of radius R is given by the inequalities

$$0 \leq \rho \leq R, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi.$$

- Points with $\phi = 0$ are on the positive z-axis.
- Points with $\phi = \pi/2$ are on the xy-plane (i.e., $z = 0$).
- Points with $\phi = \pi$ are on the negative z-axis.
- For other values of C, the equation $\phi = C$ defines a nappe of a cone.

Conversion To and From Spherical Coordinates

Conversion from spherical to Cartesian:

$$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho \sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

Conversion from Cartesian to spherical:

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \tan(\theta) = \frac{y}{x} \qquad \cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Conversion from spherical to cylindrical:

$$r = \rho\sin(\phi) \qquad \theta = \theta \qquad z = \rho\cos(\phi)$$

Conversion from cylindrical to spherical:

$$\rho = \sqrt{r^2 + z^2} \qquad \theta = \theta \qquad \cos(\phi) = \frac{z}{\sqrt{r^2 + z^2}}$$

Conversion To and From Spherical Coordinates

Example 6: Convert the spherical point $(\rho, \theta, \phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ to Cartesian coordinates.

<u>Solution:</u>

$$
\begin{aligned}
x &= \rho \sin(\phi)\cos(\theta) &&= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) &&= \frac{\sqrt{6}}{2} \\
y &= \rho \sin(\phi)\sin(\theta) &&= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) &&= \frac{\sqrt{6}}{2} \\
z &= \rho \cos(\phi) &&= 2\left(\frac{1}{2}\right) &&= 1
\end{aligned}
$$

Conversion To and From Spherical Coordinates

▸ Completing Square Review Video

Example 7: Describe the surface given by the equation in spherical coordinates

$$\rho = 4\sin(\phi)\sin(\theta) + 2\sin(\phi)\cos(\theta).$$

<u>Solution:</u> Multiply by ρ:

$$\rho^2 = 4\rho\sin(\phi)\sin(\theta) + 2\rho\sin(\phi)\cos(\theta).$$

Then convert to Cartesian coordinates:

$$x^2 + y^2 + z^2 = 4y + 2x$$

Completing the square for x and y, we obtain

$$(x-1)^2 + (y-2)^2 + z^2 = 5$$

which is a sphere.